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I. DeRham Cohomology

Def: let M be an n-mfd
we call a k-form ω on M

~~$\omega: M \to \coprod_{x \in M} Alt^k(T_xM)$~~

$\omega \in \Lambda^k(T^*M)$

$\Rightarrow$ locally: $\omega_x : (T_xM)^k \longrightarrow \mathbb{R}$

were 0-forms are just $\omega: M \to \mathbb{R}$

we call $\Omega^k(M)$ the vectorspace of differential ~~forms~~ k-forms.

$\Omega^0(M) = C^\infty(M)$

Def (wedge product) For M an n-mfd we define the wedgeproduct as

$$\wedge : \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$$
$$(\omega, \eta) \longmapsto \omega \wedge \eta$$

locally by $(\omega \wedge \eta)_x = \omega_x \wedge \eta_x$

and $f \in C^\infty(M)$: $f \wedge \eta = f\eta$

Lemma 2.8: $\Omega^*(M) := \bigoplus_{k=1}^{\infty} \Omega^k(M)$

~~is~~ a graduate unitary anticommutative Algebra ~~Λ~~

we ~~know~~ know the differentials

$$dx^i : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad dx^i\Big(\sum \lambda_j e_j\Big) = \lambda_i$$

where $dx^i \wedge dx^i = 0$, $dx^i \wedge dx^j = -dx^j \wedge dx^i$

we therefore write for $\omega \in \Omega^k(\mathbb{R}^n)$

$$\omega_x = \sum_{1 \le i_1 < \dots < i_k \le n} f_{i_1, \dots, i_k}(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

From now on: $I = (i_1, \dots, i_k)$, $|I| = \sum_{j=0}^{k} i_j$, $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$\Rightarrow \omega_x = \sum_I f_I(x)\, dx^I, \quad \eta_x = \sum_J g_J(x)\, dx^J$$

$$\omega \wedge \eta = \sum_{I, J} f_I g_J\, dx^I \wedge dx^J$$

Def (exterior Derivative) let $\omega \in \Omega^k(\mathbb{R}^n)$
we define

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

by (i) $\omega = f \in C^\infty(M) \Rightarrow d\omega = df = \sum \frac{\partial f}{\partial x^i} dx^i$
(ii) $\omega = \sum f_I\, dx^I \Rightarrow d\omega = \sum df_I \wedge dx^I$

Satz: (i) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge \eta$, $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$
(ii) $d \circ d(\omega) = 0$

Let $\omega = \sum_I f_I dx^I$, $\eta = \sum_J g_J dx^J$

(i) $d(\omega \wedge \eta) = \sum d(f_I g_J) \wedge dx^I \wedge dx^J$

$= \sum (df_I g_J + f_I dg_J) \wedge dx^I \wedge dx^J$

$\stackrel{!}{=} \sum df_I g_J \wedge dx^I \wedge dx^J + f_I \underbrace{dg_J \wedge dx^I}_{(-1)^k} \wedge dx^J$

$= d\omega \wedge \eta + (-1)\, \omega\, d\eta$

(ii) $d^2 f = d(df) = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx^i \wedge dx^j$

by $\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i} f\right)$ &

$dx^i \wedge dx^j = - dx^j \wedge dx^i$

$\Rightarrow d^2 f = 0$ for $f \in C^\infty(U)$

$d^2\omega = d\left(\sum d f_I \wedge dx^I\right)$

$\stackrel{!}{=} \sum d^2 f_I \wedge dx^I - df_I \wedge d^2 x^I = 0$ $\boxtimes$

Def: $U \subset \mathbb{R}^n$ offen

(1) $\omega \in \Omega^k(U)$ is closed, iff $d\omega = 0$

(2) $\omega \in \Omega^k(U)$ is exact, iff $\exists \eta \in \Omega^{k-1}(U)$ $d\eta = \omega$

Rem: $\mathrm{im}\{d : \Omega^{k-1}(U) \to \Omega^k(U)\} \subset \ker\{d : \Omega^k(U) \to \Omega^{k+1}(U)\}$

For Manifolds Let $f: M \to N$

$\Rightarrow \exists f^* := \Omega^* f : \Omega^*(N) \to \Omega^*(M)$ linear

Def: $(f^*\omega)_x (v_1, \dots, v_k) := \omega_{f(x)} (df_x v_1, \dots, df_x v_k)$

Lemma: (i) $f^*(\lambda \omega_1 + \mu \omega_2) = \lambda f^*\omega_1 + \mu f^*\omega_2$

(ii) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

(iii) $f^*(d\omega) = d(f^*\omega)$, $\omega \in \Omega^k(U)$

Bew: (iii) $d(f^*\phi) = \sum \frac{\partial(\phi \circ f)}{\partial x_\lambda} dx^\lambda$

$\stackrel{!}{=} \sum_{\lambda=1}^{n} \sum_{\mu=1}^{m} \left(\frac{\partial \phi}{\partial y_\mu} \circ f\right) \frac{\partial f_\mu}{\partial x_\lambda} dx_\lambda$

$\stackrel{!}{=} \sum \left(\frac{\partial \phi}{\partial y_\mu} \circ f\right) d f_\mu$

$\stackrel{!}{=} f^*(d\phi)$

analogous to previous, this holds for k-forms.

Def (DeRham Kohomology)

$$H^k_{dR}(U) = \frac{\ker\{d: \Omega^k(U) \to \Omega^{k+1}(U)\}}{\mathrm{Im}\{d: \Omega^{k-1}(U) \to \Omega^k(U)\}}$$

$\Rightarrow \omega \in \Omega^k(U) \rightsquigarrow [\omega] \in H^k_{dR}(U)$

$[\omega] = \omega + d\,\Omega^{k-1}(U)$

Satz: H^* ist ein Funktor, Invariant unter Diffeomorphismen

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<u>what we skip</u>

This Notion is just working for $\mathbb{R}^n$ and open ~~[illegible]~~ Neighbourhoods. But via the

- Mayer - Vietoris sequence

and

- k-forms ~~w/~~ compact support

and

- Partition of union, ~~we translate~~

we can transfere this notion to an arbitrary C^∞-Manifold.

- Homotopy invariance / Poincare-Lemma

<u>what we do</u> (Homotopy Invariance)

let : $F, G: M \to N$ s.t.

$\exists H: M \times I \to N$, $H(x,0) = F(x)$, $H(x,1) = G(x)$

$F^* = G^* : H^*_{dR}(M) \to H^*_{dR}(N)$

therefore show: ~~[illegible]~~ $[\omega] \in H^k_{dR}(M)$

$F^*\omega - G^* \omega = d\eta$, $\eta \in H^{k-1}(M)$

<u>Goal</u>: define $h: \Omega^k(N) \to \Omega^{k-1}(M)$ s.t.

$d(h\omega) + h(d\omega) = G^*\omega - F^*\omega$

by

$i_t: M \to M \times I$

<u>Lemma</u>: $i_0^*, i_1^* : \Omega^*(M \times I) \to \Omega^*(M)$ is ~~a homotopy operator.~~

<u>THM</u> (HOMOTOPY INVARIANCE)

M, N homotopy equivalent $\Rightarrow H^p_{dR}(M) \cong H^p_{dR}(N)$

II SMOOTH SINGULAR HOMOLOGY

~~Recall~~ $\Delta_k := [e_0, \dots, e_k] \subset \mathbb{R}^m$ a singular k-simplex

we call $f \in C^\infty(\Delta_k, M)$ a smooth k-simplex

s.t. $C^\infty_k(M) = \{ \sigma: \Delta_k \to M \mid \sigma \text{ is smooth} \}$

is the free abelian group generated by smooth k-simplices. we can so

$$C^\infty_k(M) \subset C_k(M)$$

so w/ proof.

$$H^\infty_k(M) = \frac{\ker\{\partial: C^\infty_k(M) \to C^\infty_{k-1}(M)\}}{\operatorname{Im}\{\partial: C^\infty_{k+1}(M) \to C^\infty_k(M)\}}$$

is called smooth singular Homology.

$\partial\sigma = \sum_{n=0}^{k} (-1)^n \sigma|_{[v_0, \dots, \hat{v}_n, \dots, v_n]}$ is smooth for σ smooth

Lets look at

$$i : C_k^\infty(M) \longrightarrow C_k(M)$$

we can see:

$$i \circ \partial = \partial \circ i$$

and it induces

$$i_* : H_k^\infty(M) \longrightarrow H_k(M)$$

by $i_*[\omega] = [i\omega]$

<u>Thm</u> For M a C^∞-Mfd

$$i_* : H_k^\infty(M) \longrightarrow H_k(M)$$

is an Isomorphismus

II deRhams Theorem

To talk about deRhams Theorem, we firstly need to talk about integration:

consider $\omega \in \Omega^k(M)$,

$T : \Delta_k \longrightarrow M$ smooth

we define

$$\int_T \omega := \int_{\Delta_k} T^* \omega$$

naturally: $c = \sum c_i T_i \in C_k^\infty(M)$ (chain)

$$\int_c \omega := \sum_i c_i \int_{T_i} \omega$$

<u>Thm</u> (Stokes) $c \in C_k^\infty(M)$, $\omega \in \Omega^{k-1}(M)$

$$\int_{\partial c} \omega = \int_c d\omega$$

This yields the following natural linear map:

$$\mathcal{l} : H_{dR}^k(M) \longrightarrow H^k(M; \mathbb{R})$$

by $H^k(M;\mathbb{R}) \cong \mathrm{Hom}(H_k(M);\mathbb{R})$

$$\mathcal{l}[\omega][c] = \int_c \omega$$

this is well defined by

$c, c' \in [c] \Rightarrow c - c' = \partial c''$

$$\Rightarrow \int_c \omega - \int_{c'} \omega = \int_{c''} d\omega = 0$$

as well as for $\omega = d\eta$

$$\int_c \omega = \int_c d\eta = \int_{\partial c} \eta = 0$$

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<u>Prop</u>: The deRham-Map $\mathcal{I}$ fulfills:

(a)
$$\begin{tikzcd} H^k_{dR}(N) \arrow[r, "f^*"] \arrow[d, "\mathcal{I}"'] & H^k_{dR}(M) \arrow[d, "\mathcal{I}"] \\ H^k(N;\mathbb{R}) \arrow[r, "f^*"] & H^k(M;\mathbb{R}) \end{tikzcd}$$

(b)
$$\begin{tikzcd} H^{k-1}_{dR}(U\cap V) \arrow[r, "\delta"] \arrow[d, "\mathcal{I}"'] & H^k_{dR}(M) \arrow[d, "\mathcal{I}"] \\ H^{k-1}(U\cap V;\mathbb{R}) \arrow[r, "\xi"] & H^{k}(M;\mathbb{R}) \end{tikzcd}$$

where δ, ξ the maps from the Mayer Vietoris sequence.

<u>Proof</u>: σ a smooth k-simplex
$\omega \in \Omega^k(N)$

$$\mathcal{I}(f^*[\omega])([\sigma]) = \int_{\Delta_k} \sigma^* f^* \omega$$

$$= \int_{\Delta_k} (f\circ\sigma)^*\omega = \int_{f\circ\sigma} \omega = \mathcal{I}([\omega])\circ f([\sigma])$$

$$= f^*(\mathcal{I}([\omega]))([\sigma])$$ + linear ext.

we get closer to the main theorem.

A smooth-Manifold M is called deRham, iff
$$\mathcal{I} : H^k_{dR}(M) \to H^k(M;\mathbb{R})$$
is an Isomorphism.

An open cover $\{U_i\}_{i\in I}$ is called deRham if all finite intersections are deRham.

<u>THM</u> (deRham) Every smooth Manifold is deRham.

<u>Proof</u>. 1 Every mfd w/ a finite deRham Cover is dR

Induction: for $M=U$ trivial ☑

Suppose $M = U\cup V$:

$$\begin{tikzcd}[column sep=small] H^{k-1}_{dR}(U)\oplus H^{k-1}_{dR}(V) \arrow[r] \arrow[d, "\mathcal{I}\oplus\mathcal{I}"] \arrow[dr, phantom, "1"] & H^{k-1}_{dR}(U\cap V) \arrow[r] \arrow[d, "\mathcal{I}"] \arrow[dr, phantom, "2"] & H^k_{dR}(M) \arrow[r] \arrow[d, "\mathcal{I}"] \arrow[dr, phantom, "3"] & H^k_{dR}(U)\oplus H^k_{dR}(V) \arrow[r] \arrow[d, "\mathcal{I}\oplus\mathcal{I}"] \arrow[dr, phantom, "4"] & H^k_{dR}(U\cap V) \arrow[d, "\mathcal{I}"] \\ H^k(U;\mathbb{R})\oplus H^k(V;\mathbb{R}) \arrow[r] & H^k(U\cap V;\mathbb{R}) \arrow[r] & H^k(M;\mathbb{R}) \arrow[r] & H^k(U;\mathbb{R})\oplus H^k(V;\mathbb{R}) \arrow[r] & H^k(U\cap V;\mathbb{R}) \end{tikzcd}$$

Proposition gives us commutativity. $U, V, U\cup V, U\cap V$ are all deRham. therefore from 5-Lemma and Induction Hypothesis $H^k_{dR}(M) \cong H^k(M;\mathbb{R})$ is iso, by $V = U_N$ [illegible]

(2) M has deRham Basis $\{U_i\}_{i\in I}$

take a C^∞-positive exhaustion function

$f: M \to \mathbb{R}$, s.t. $f^{-1}(]-\infty, c])$ cpt.

$l \in \mathbb{N}$: $A_l := f^{-1}([l, l+1])$

$\Rightarrow M = \bigcup_{l\in\mathbb{N}} A_l$

define $A_l' := f^{-1}(]l-\frac{1}{2}, l+\frac{3}{2}[)$

$\Rightarrow A_l \subset A_l'$ and

A_l' can be written as Union of

basis open sets U_i

they cover the cpt. A_l, so choose

a finite subcover $(U_{i_k})_{k=1,\dots,n}$

and def: $B_l = \bigcup_{k=1}^{n} U_{i_k}$

we know that B_l is deRham.

From $B_l \subseteq A_l'$,

$U := \bigcup_{l \text{ odd}} B_l$, $V := \bigcup_{l \text{ even}} B_l$

Are each disjoint unions of DeRham sets

~~i_j~~ $I_j : B_j \to \bigsqcup_{j\in J} B_j$ the inclusion

induces an Iso between direct product of

cohomology groups and cohomology of the

disjoint Union.

From the commutativity we see, that each

U and V are DeRham. Therefore

$\{U, V\}$ is a finite DeRham Cover of M

$\Rightarrow M$ is deRham

Finally: Every Convex set U is deRham, by

the Poincaré lemma: $\dim H^k_{dR}(U) = 1$ for $k=0$

and trivial for $k>0$.

Analogous $H^k(U;\mathbb{R}) = \mathrm{Hom}(H_k(U);\mathbb{R})$ is gen.

by any dual of $\sigma: \Delta_0 = \{0\} \to M$. for $k=0$

$\Rightarrow \mathcal{I}[\mathbb{1}]([\sigma]) = \int_{\Delta_0} \sigma^* \mathbb{1} = (\mathbb{1}\circ\sigma)(0) = 1$

is nontrivial and therefore ~~the~~ an isomorphism.

secondly all $U \subset \mathbb{R}^n$ open are deRham.

The euklidean Balls give a countable $\overset{\text{convex}}{\text{basis}}$

And every Manifold has a basis of Atlas

Charts where each intersection is diffeomorphic

to an open set $U \subset \mathbb{R}^n$.

$\Rightarrow M$ is deRham